Final Exam

CS 251, Winter 2018
Discrete Structures II
David Lu

Name ________________________________

March 15^{th}, 2018

This is a closed-book (no notes, cellphones, smart watches, etc...) exam. Show your work for partial credit. Read the questions carefully. Write legibly and document your proof steps. Feel free to use the back of each page if you need more room for work.

Part 1: True/False (3 pts each; 15 pts total)

True False A) Resolution can be used to show that an argument is valid.
Yes. The last page asks you to do this.

True **False** B) Two interpretations are necessary to show that two sentences are logically equivalent.
No. Showing two sentences are logically equivalent requires two proofs.

True False C) All subproofs must be discharged before a proof can be considered complete.
Yes. Proof is not done if subproofs are still open.n

True False D) $\forall x \forall y((Ax \land Ay) \to Rxy)$ and $\forall y \forall x((Ax \land Ay) \to Rxy)$ are logically equivalent.
Recall that the positions of same quantifiers can be exchanged if they are next to each other. Positions of different quantifiers cannot be.

True False E) If a and b are identical and Pa is true, then Pb is true.
If two constants name the same thing, then any predicates true of one constant must be true of the other.

Part 2: Translations Translate the sentences to FOL (5 pts each; 15 pts total)
A) Everyone who studies Computer Science has a friend who can program.
(Cx: x studies computer science, Fxy: x is y's friend, Px: x can program)

$\forall x(Cx \to \exists y(Fyx \land Py))$
Notice the order of the variables for the friend predicate, since the student of CS *has a friend* who can program.

B) If there are at least three tarantulas in the car, then I'm not getting in the car.
(Tx: x is a tarantula, Cx: x is in the car, Gx: x is getting in the car.)

$\exists x \exists y \exists z(Tx \land Cx \land Ty \land Cy \land Tz \land Cz \land \neg x = y \land \neg x = z \land \neg y = z) \to Gi$
Notice structure of this sentence. It is a conditional sentence not an existential sentence. Conditional is the main connective.

C) Bob doesn't love himself, but someone loves him.
(Lxy: x loves y)
$\neg Lbb \land \exists x Lxb$

Part 3: Tautology, Contingency, Contradiction Determine whether the formula is a *tautology*, *contingency*, or *contradiction*, then demonstrate your answer with the appropriate method (10 pts each; 30 pts total)

A) $\forall x((Px \land Qx) \to Rx) \to [(\forall xPx \land \forall xQx) \to \forall xRx]$

Tautology. The sentence says "If anything thats both a P and a Q is an R, then if everything is a P and everything is a Q, then everything is an R." Has to be true in any interpretation. Proof:

1.	$\forall x((Px \land Qx) \to Rx)$	for →I
2.	$\forall xPx \land \forall xQx$	for →I
3.	$\forall xPx$	∧E
4.	$\forall xQx$	∧E
5.	Pa	∀E, 3
6.	Qa	∀E, 4
7.	$(Pa \land Qa) \to Ra$	∀E, 1
8.	$Pa \land Qa$	∧I 5, 6
9.	Ra	→E 7, 8
10.	$\forall xRx$	∀I 9
11.	$(\forall xPx \land \forall xQx) \to \forall xRx$	→I 2-10
12.	$\forall x((Px \land Qx) \to Rx) \to [(\forall xPx \land \forall xQx) \to \forall xRx]$	→I 1-11

B) $\forall x \exists y Rxy \leftrightarrow \forall x \exists y Ryx$

Contingent. HW assignment chapter 30, part B 9 and we did it in class. Left side says that everything is R-related to something. Right side says that everything has something that's R-related to it. So find an interpretation that makes both sides true to make the biconditional true. And find an interpretation that makes one side true but the other side false, making the biconditional false.

True:
$D = \{a\}$
Raa

False:
$D = \{a, b\}$
$Raa, \neg Rab, Rba, \neg Rbb$

C) $\forall x \forall y((Px \land Py) \to x = y)$

Contingent. This sentence says that any things that are P are the same thing. So find an interpretation where all Ps are identical. The sentence is true at this interpretation. And find an interpretation where not all Ps are identical. That would be an interpretation where the sentence is false.

True:
$D = \{a\}$
Pa

False:
$D = \{a, b\}$
$Pa, Pb, \neg a = b$

Part 4: Validity and Invalidity Determine if the following arguments are valid or not. Demonstrate with appropriate method. (15 pts each; 30 pts total)

A) $\neg\exists x(Px \wedge Qx), \exists xPx \therefore \neg\forall x(Px \rightarrow Qx)$

Valid. The argument goes "There does not exist anything that's both P and Q. There is a P. Therefore, it's not the case that all Ps are Qs." That's valid. The premises tell us that nothing is both P and Q and that there is a P. So that thing which is a P must not be a Q. So not all Ps are Qs. Proof:

1.	$\neg\exists x(Px \wedge Qx)$	Premise
2.	$\exists xPx$	Premise
3.	$\forall x(Px \rightarrow Qx)$	for ¬ I
4.	Pa	for ∃E 2
5.	$Pa \rightarrow Qa$	∀E 3
6.	Qa	→E 5, 4
7.	$Pa \wedge Qa$	∧I 4, 6
8.	$\exists x(Px \wedge Qx)$	∃I 7
9.	$\bot$	¬E 1, 8
10.	$\bot$	∃E 2, 4-9
11.	$\neg\forall x(Px \rightarrow Qx)$	¬ I 3 - 10

B) $\exists x\forall yPxy$, $\forall x(Pxx \rightarrow \exists yQyx))$, $\therefore \exists y\exists xQxy$ (Hint: This is valid)

As hinted, this is valid. Just be careful with the rules and do it step by step and remember that ∃E is a subproof rule.

1.	$\exists x\forall yPxy$	Premise
2.	$\forall x(Pxx \rightarrow \exists yQyx)$	Premise
3.	$\forall yPay$	for ∃E 1
4.	Paa	∀E 3
5.	$Paa \rightarrow \exists yQya$	∀E 2
6.	$\exists yQya$	→E 5, 4
7.	Qba	for ∃E 6
8.	$\exists xQxa$	∃I 7
9.	$\exists y\exists xQxy$	∃I 8
10.	$\exists y\exists xQxy$	∃E 6, 7-9
11.	$\exists y\exists xQxy$	∃E 1, 3-10

Part 5: Resolution (10 pts)

A) Demonstrate that the following argument is valid using resolution.
$A \to (B \to C) \therefore (A \wedge B) \to C$

Algorithm to show argument is valid by resolution: Negate the conclusion. Turn all premises and conclusion into conjunctive normal form. Take all of the disjunctive clauses and iterate resolution until you reach the empty clause. Remember that the goal in all resolutions proofs is to reach the empty clause.

Notice that the premise is equivalent to the conclusion. So they have the same truth table. However, they will have opposite truth tables since we want the clauses from the negated conclusion. So, since we have 3 atoms, we need 8 rows on the truth table; and between the two sentences there will be 8 false rows, we should have 8 clauses total.

A	B	C	**Premise** $A \to (B \to C)$	**Negated Conclusion** $\neg((A \wedge B) \to C)$
T	T	T	T	F
T	T	F	F	T
T	F	T	T	F
T	F	F	T	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	T	F

Clause from premise: $\{\neg A \vee \neg B \vee C\}$

Clauses from conclusion: $\{\neg A \vee \neg B \vee \neg C\}$, $\{\neg A \vee B \vee \neg C\}$, $\{\neg A \vee B \vee C\}$, $\{A \vee \neg B \vee \neg C\}$, $\{A \vee \neg B \vee C\}$, $\{A \vee B \vee \neg C\}$, $\{A \vee B \vee C.\}$

1.	$\neg A \vee \neg B \vee C$	Premise
2.	$\neg A \vee \neg B \vee \neg C$	Premise
3.	$\neg A \vee B \vee \neg C$	Premise
4.	$\neg A \vee B \vee C$	Premise
5.	$A \vee \neg B \vee \neg C$	Premise
6.	$A \vee \neg B \vee C$	Premise
7.	$A \vee B \vee \neg C$	Premise
8.	$A \vee B \vee C$	Premise
9.	$\neg A \vee \neg B$	Reso 1, 2
10.	$\neg A \vee B$	Reso 3, 4
11.	$\neg A$	Reso 9, 10
12.	$A \vee \neg B$	Reso 5, 6
13.	$\neg B$	Reso 11, 12
14.	$A \vee B$	Reso 7, 8
15.	A	Reso 13, 14
16.	$\square$	11, 15

——Proof is much shorter if you picked up the separate rule set from the resolution reading. ——
$\neg((A \wedge B) \to C)$ (Negated conclusion)
$= \neg(\neg(A \wedge B) \vee C)$ (I)
$= \neg\neg(A \wedge B) \wedge \neg C$ (N)
$= A \wedge B \wedge \neg C$ (N) (–Notice this is CNF and each conjunct is a clause.)

1.	$\neg A \vee \neg B \vee C$	Premise
2.	A	Premise
3.	B	Premise
4.	$\neg C$	Premise
5.	$\neg B \vee C$	Reso 1, 2
6.	C	Reso 3, 5
7.	$\square$	Reso 4, 6